

How can we know that a knot cannot be undone?

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Failing to untie a tangled cord does not prove that it is impossible. Perhaps we have not found the right sequence of movements. Mathematics answers a sharper question by specifying the object and the permitted transformations, then identifying a property that every permitted transformation preserves.

The standard knot-theoretic object is a closed loop embedded in three-dimensional space. It has no loose end that can simply be pulled through. We may deform it without cutting the loop or passing one strand through another. Whether it can become an unknotted circle is then a question about its topology, not about patience, friction or the strength of one's fingers. [1]

1. Why closing the ends changes the question

An ordinary shoelace has ends and thickness. Pulling an end through a loop is allowed in many practical untangling tasks; it is not an operation on a mathematical closed knot. Fixing endpoints, joining them or allowing them to move creates different problems. A claim of impossibility must specify which one it addresses.

The mathematical model also ignores stretching costs and material failure. A physically jammed configuration can be difficult to manipulate without being topologically nontrivial. Conversely, a closed loop can remain knotted even when it is perfectly flexible and frictionless. Mechanical obstruction and topological obstruction should not be treated as synonyms.

We will consider ordinary smooth or polygonal closed knots, rather than pathological embeddings with infinitely complicated local structure. That scope is enough for the diagram-based argument below and makes the permitted manipulations well defined.

2. A drawing needs crossing information

A flat projection of a knot contains apparent crossings. At each crossing we must record which strand passes over the other. Without that information, the drawing does not determine the spatial arrangement it is meant to represent.

A crossing in a diagram is not a physical intersection of the two strands. They are separated in the third dimension. Moving the loop can create or remove crossings in its projection, so simply counting crossings in one drawing does not prove that the loop is knotted.

Reidemeister moves describe three local changes of knot diagrams: creating or removing a curl, creating or removing a pair of crossings, and sliding a strand past a crossing. For the class of knots considered here, equivalent diagrams can be connected by these moves, together with planar deformation. [1] This provides a finite vocabulary for checking whether a diagram property survives every permitted change.

3. Colour as a test of structure

Divide a diagram into arcs that run from one undercrossing to the next. Assign each arc one of three colours. At every crossing, require either all three incident arc colours to be the same or all three to be different. A nontrivial colouring uses more than one colour somewhere in the diagram.

The trefoil has a diagram with three arcs that can be assigned three different colours while satisfying every crossing. An unknotted circle in its crossing-free diagram has just one arc, so it has no nontrivial colouring. The useful result is that existence of a nontrivial three-colouring is preserved by Reidemeister moves. [1]

If that preservation holds, the conclusion follows: a trefoil cannot become a crossing-free circle through the permitted deformations. Otherwise a sequence of moves would preserve nontrivial colourability at every step and arrive at a diagram that lacks it. That is a contradiction.

4. Why the colouring rule can survive a move

There is an algebraic way to inspect the local rule. Label colours 0, 1 and 2, and calculate modulo three. If the overpassing arc has colour a and the two underpassing arcs have colours b and c , the condition is $2a = b + c$ modulo three. Equal colours satisfy it; three distinct colours also satisfy it.

Given a and b , the continuation colour is uniquely $c = 2a - b$ modulo three. Applying that operation twice with the same overpassing colour returns b , since $2a - (2a - b) = b$. This is the algebra behind reversibility when a cancelling pair of crossings is inserted or removed.

For the three-strand move, write $f_a(b) = 2a - b$. The relevant compatibility is $f_a(f_b(c)) = f_{f_a(b)}(f_a(c))$. Both sides reduce to $2a - 2b + c$ modulo three. For a curl, $f_a(a) = a$. These relations give the local consistency needed for colourings to extend and retract through the three moves. They also preserve the all-one-colour assignments, so a nontrivial colouring cannot disappear by being secretly converted into a trivial one.

The algebra is valuable because it replaces trust in a complicated sketch with equations one can check. A diagrammatic proof still has to associate the equations with the moves and their strands; the symbols do not remove that representational duty.

5. A test that succeeds is not a complete classifier

If two diagrams have different values of a knot invariant, they cannot represent the same knot. If they agree, the invariant may simply be too coarse to distinguish them. Nontrivial three-colourability supplies an obstruction to unknotting, not a promise to resolve every knot.

This asymmetry appears in many mathematical tests. A preserved property can rule out a proposed transformation while failing to identify a unique object. More powerful invariants and algorithms address broader questions, but adding their names is not necessary to understand the trefoil argument.

The certainty here is conditional on the model and proof. It does not imply that every knot in a real cord is physically impossible to undo, because open ends or cutting may change the permitted operations. Nor does unsuccessfully trying many diagrams establish a topological obstruction.

We can know that a particular closed knot cannot be undone by finding a property it has, proving that every allowed deformation preserves that property, and showing that the

unknot lacks it. The decisive step is preservation across all allowed moves. That is what turns a frustrating failure to find a solution into a mathematical reason that no such solution exists.

References

[1] Hoberg, R. Knots and Braids. University of Chicago REU paper, 26 August 2011, §3.
<https://math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Hoberg.pdf>

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